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Forward and Converse Theorems of Polynomial Approximation for Exponential Weights on $[-1, 1]$, II

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We consider exponential weights of the form $w := e^{-Q}$ on $[-1, 1]$ where $Q(x)$ is even and grows faster than $(1-x^2)^{-\delta}$ near ± 1, some $\delta > 0$. For example, we can take

$$Q(x) := \exp_k((1-x^2)^{-\alpha}), \qquad k \geqslant 0, \alpha > 0,$$

where $\exp_k$ denotes the kth iterated exponential and $\exp_0(x) = x$. We prove converse theorems of polynomial approximation in weighted L_p spaces with norm $\|fw\|_{L_p[-1,1]}$ for all $0 < p \leqslant \infty$, to match the forward theorems proved in part I of this paper.

1. STATEMENT OF RESULTS

There is a well developed theory of weighted polynomial approximation for weights $w: (-1, 1) \to (0, \infty)$ that behave like Jacobi weights near ± 1 [9]. However, for weights that decay rapidly near ± 1, this theory does not apply. In this paper, we prove converse theorems of polynomial approximation for even weights

$$w := e^{-Q} \tag{1.1}$$

where $Q: (-1, 1) \to \mathbb{R}$ is even and grows at least as fast as $(1-x^2)^{-\delta}$, $\delta > 0$, near ± 1. In part I of this paper [16], we proved Jackson theorems for these weights; that is, we estimated

$$E_n[f]_{w,p} := \inf_{P \in \mathcal{P}_n} \|(f-P)\, w\|_{L_p[-1,1]}, \tag{1.2}$$

$0 < p \leqslant \infty$, where $\mathcal{P}_n$ denote the polynomials of degree at most n.

Our methods are similar to those in [8], where Jackson theorems were proved for Freud weights, and to the follow up papers [2, 3] where Erdös

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weights were treated. To state our results, we need to define our class of weights, as well as various quantities. First, we say that a function $f: (a, b) \to (0, \infty)$ is *quasi-increasing* if $\exists C > 0$ such that

$$a < x < y < b \Rightarrow f(x) \leqslant Cf(y).$$

DEFINITION 1.1. Let $w := e^{-Q}$, where

(a) $Q: (-1, 1) \to \mathbb{R}$ is even, continuous, has limit ∞ at 1, and Q' is positive in $(0, 1)$.

(b) $xQ'(x)$ is strictly increasing in $(0, 1)$ with right limit 0 at 0.

(c) The function

$$T(x) := \frac{Q'(x)}{Q(x)} \tag{1.3}$$

is quasi-increasing in $(C, 1)$ for some $0 < C < 1$.

(d) $\exists\, C_1, C_2, C_3 > 0$ such that

$$\frac{Q'(y)}{Q'(x)} \leqslant C_1 \left(\frac{Q(y)}{Q(x)}\right)^{C_2}, \qquad y \geqslant x \geqslant C_3. \tag{1.4}$$

(e) For some $\delta > 0$, $0 < C < 1$, $(1 - x^2)^{1+\delta}\, Q'(x)$ is increasing in $(C, 1)$. Then we write $w = e^{-Q} \in \mathcal{E}$.

The archetypal example of $w \in \mathcal{E}$ is

$$w(x) := w_{k,\alpha}(x) := \exp(-\exp_k([1 - x^2]^{-\alpha})), \qquad k \geqslant 0, \quad \alpha > 0, \tag{1.5}$$

where $\exp_k = \exp(\exp(\cdots))$ denotes the kth iterated exponential and $\exp_0(x) = x$ [16].

We need the condition that $xQ'(x)$ be strictly increasing to guarantee the existence of the *Mhaskar–Rahmanov–Saff number* a_u, the positive root of the equation

$$u = \frac{2}{\pi} \int_0^1 a_u t Q'(a_u t) \frac{dt}{\sqrt{1 - t^2}}, \qquad u > 0. \tag{1.6}$$

For those to whom a_u is new, its significance lies partly in the identity [19–21]

$$\|Pw\|_{L_\infty[-1, 1]} = \|Pw\|_{L_\infty[-a_n, a_n]}, \qquad P \in \mathcal{P}_n \tag{1.7}$$

and the fact that a_n is the "smallest" such number.

Our modulus of continuity involves two parts, a "main part" and a "tail." The main part involves rth symmetric differences over the interval $[-a_{1/(2t)}, a_{1/(2t)}]$, and the tail involves an error of weighted polynomial approximation over the remainder of $(-1, 1)$. For $h>0$, an interval J, and $r\geq 1$, we define the rth symmetric difference

$$\Delta_h^r(f, x, J) := \sum_{i=0}^{r} \binom{r}{i} (-1)^i f\left(x+\frac{rh}{2}-ih\right), \tag{1.8}$$

provided all arguments of f lie in J, and 0 otherwise. Sometimes, we just write $\Delta_h^r f(x)$ if it is clear which interval J we are using. Sometimes the increment h will depend on x and the function

$$\Phi_t(x) := \sqrt{\left|1-\frac{|x|}{a_{1/t}}\right|} + T(a_{1/t})^{-1/2}, \qquad x\in(-1, 1). \tag{1.9}$$

This is the case in our modulus of continuity

$$\omega_{r,p}(f, w, t) := \sup_{0<h\leq t} \|w\Delta_{h\Phi_t(x)}^r(f, x, (-1, 1))\|_{L_p(|x|\leq a_{1/(2t)})} + \inf_{P\in\mathscr{P}_{r-1}} \|(f-P)\, w\|_{L_p(a_{1/(4t)}\leq |x|\leq 1)} \tag{1.10}$$

and its averaged cousin

$$\bar{\omega}_{r,p}(f, w, t) := \left[\frac{1}{t}\int_0^t \|w\Delta_{h\Phi_t(x)}^r(f, x, (-1, 1))\|_{L_p(|x|\leq a_{1/(2t)})}^p\, dh\right]^{1/p} + \inf_{P\in\mathscr{P}_{r-1}} \|(f-P)\, w\|_{L_p(a_{1/(4t)}\leq |x|\leq 1)}. \tag{1.11}$$

(If $p=\infty$, we set $\bar{\omega}_{r,p} := \omega_{r,p}$.) See [16] for further discussion of the modulus. Here we simply note that the function $\Phi_t(x)$ describes the improvement in the degree of approximation near $\pm a_n$, in much the same way that $\sqrt{1-x^2}$ does for Jacobi weights on $[-1, 1]$. The main result of part I of this paper [16] is

THEOREM 1.2. *Let $w := e^{-Q}\in\mathscr{E}$. Let $r\geq 1$ and $0<p\leq\infty$. Then for $f\colon(-1, 1)\to\mathbb{R}$ for which $fw\in L_p(-1, 1)$ (and for $p=\infty$, we require f to be continuous and fw to vanish at ± 1), we have for $n\geq C_3$*

$$E_n[f]_{w,p} \leq C_1\bar{\omega}_{r,p}\left(f, w, \frac{C_2}{n}\right) \leq C_1\omega_{r,p}\left(f, w, \frac{C_2}{n}\right), \tag{1.12}$$

where C_j, $j=1,2,3$, *do not depend on* f *or* n. *Moreover*,

$$E_n[f]_{w,p} \leqslant C_1 \inf_{\rho \in [3/4,1]} \bar{\omega}_{r,p}\left(f, w, C_2 \frac{\rho}{n}\right). \tag{1.13}$$

(The inequality (1.13) was stated as Theorem 6.3 in [16].)

In establishing converse theorems of polynomial approximation, the K-functional plays a crucial role [9]. In our context, a suitable K-functional is

$$K_{r,p}(f, w, t^r) := \inf_g \{ \|(f-g)\, w\|_{L_p[-1,1]} + t^r \, \|g^{(r)} w \Phi_t^r\|_{L_p[-1,1]} \}, \tag{1.14}$$

where the inf is taken over all g whose $(r-1)$st derivative is locally absolutely continuous. The presence of the function Φ_t reflects "endpoint influences." Unfortunately, the K-functional is useful only for $p \geqslant 1$ as it often vanishes identically for $p<1$ [6]. So several authors have used the realisation functional, which works for all $0<p\leqslant\infty$, though it is not as elegant as the K-functional [3, 6, 8, 12]. In our context, a suitable realisation functional is

$$\bar{K}_{r,p}(f, w, t^r) := \inf_{\deg(P) \leqslant 1/t} \{ \|(f-P)\, w\|_{L_p[-1,1]} + t^r \, \|P^{(r)} w \Phi_t^r\|_{L_p[-1,1]} \}. \tag{1.15}$$

Our first result concerns the equivalence between the realisation functional and the modulus of continuity:

THEOREM 1.3. *Let* $w := e^{-Q} \in \mathscr{E}$. *Let* $r \geqslant 1$, $M>0$ *and* $0<p\leqslant\infty$. *Assume, moreover, that there is a Markov–Bernstein inequality of the form*

$$\|P_n' w \Phi_{1/n}\|_{L_p[-1,1]} \leqslant Cn \, \|P_n w\|_{L_p[-1,1]}, \qquad P_n \in \mathscr{P}_n, \tag{1.16}$$

where $C \neq C(n, P_n)$. *Then there exists* $t_0>0$ *and* $C_j>0$, $j=1,2,3$, *independent of* f, t *such that for* $t \in (0, t_0)$,

$$C_1 \omega_{r,p}(f, w, t) \leqslant \bar{K}_{r,p}(f, w, t^r) \leqslant C_2 \omega_{r,p}(f, w, t) \tag{1.17}$$

and

$$C_1 \bar{\omega}_{r,p}(f, w, t) \leqslant \bar{K}_{r,p}(f, w, t^r) \leqslant C_2 \bar{\omega}_{r,p}(f, w, t). \tag{1.18}$$

Moreover,

$$\omega_{r,p}(f, w, Mt) \leqslant C_3 \omega_{r,p}(f, w, t). \tag{1.19}$$

For $p=\infty$, the Markov–Bernstein inequality (1.16) was established in [18] under additional conditions on Q, namely, conditions on Q'', which are certainly satisfied for $w_{k,\alpha}$ of (1.5). The methods of [15, 17] and the Christoffel function estimates of [14] enable one to extend (1.16) to all $0<p\leqslant\infty$ under the conditions of [18]. Moreover, we are certain that (1.16) is true under scarcely more than we need for $w\in\mathscr{E}$.

Note too that for $p\geqslant 1$, the methods of [9] should enable one to avoid assuming the Markov–Bernstein inequality for Theorem 1.3. However, one needs to assume this inequality for the corollaries below, so we do not devote any attention to this point.

The inequality (1.19) allows us to simplify the Jackson theorem:

COROLLARY 1.4. *Under the hypotheses of Theorem* 1.3, *for* $n\geqslant C_2$

$$E_n[f]_{w,p}\leqslant C_1\bar{\omega}_{r,p}\left(f, w, \frac{1}{n}\right)\leqslant C_1\omega_{r,p}\left(f, w, \frac{1}{n}\right), \tag{1.20}$$

where C_1, C_2 *do not depend on* f *or* n.

From Theorem 1.3 we can also deduce converse theorems of approximation. For the statement, we need $\langle x\rangle$, the greatest integer $\leqslant x$.

COROLLARY 1.5. *Assume the hypotheses of Theorem* 1.3. *Let* $q:=\min\{1,p\}$. *There exists* t_0 *depending only on* w, r, p *such that for* $t\in(0,t_0)$, *and* $l:=\langle\log_2(1/t)\rangle$, *we have*

$$\omega_{r,p}(f,w,t)^q\leqslant Ct^{rq}\sum_{j=-1}^{l}(l-j+1)^{rq/2}\,2^{jrq}E_{2^j}[f]^q_{w,p}, \tag{1.21}$$

where $C\neq C(f,w,t)$ *and we set* $E_{2^{-1}}:=E_0$.

One can deduce from (1.20) and (1.21) that for $0<\alpha<r$,

$$\omega_{r,p}(f,w,t)=O(t^\alpha)\Leftrightarrow E_n[f]_{w,p}=O(n^{-\alpha}). \tag{1.22}$$

Finally, we note that for $p\geqslant 1$, the modulus may be estimated in terms of $f^{(r)}$, and the K-functional and realisation functional are equivalent:

COROLLARY 1.6. *Let* $1\leqslant p\leqslant\infty$. *Let* $w\in\mathscr{E}$.

(a) *If* $f^{(r)}w\in L_p[-1,1]$, *we have*

$$\omega_{r,p}(f,w,t)\leqslant C_1t^r\,\|f^{(r)}w\Phi_t^r\|_{L_p[-1,1]} \tag{1.23}$$

for $t\in(0,C_2)$. *Here* $C_j\neq C_j(f,t)$, $j=1,2$.

(b) *If also the Markov–Bernstein inequality* (1.16) *holds, then*

$$1 \leqslant \bar{K}_{r,p}(f, w, t^r)/K_{r,p}(f, w, t^r) \leqslant C_3, \tag{1.24}$$

for $t \in (0, C_4)$. *Here* $C_j \neq C_j(f, t)$, $j = 3, 4$.

The paper is organised as follows: In Section 2, we present some technical details related to Q, a_u, and so on. In Section 3, we present a crucial inequality that is proved using ideas of Z. Ditzian, following his proof in [8]. In Section 4, we prove Theorem 1.3. In Section 5, we prove Corollaries 1.4 to 1.6.

We close this section with more notation. Throughout, $C, C_1, C_2, \ldots$ denote positive constants independent of n, x, and $P \in \mathscr{P}_n$. The same symbol does not necessarily denote the same constant in different occurrences. We write $C \neq C(L)$ to indicate that C is independent of L. Moreover, when dealing with, for example, $x, y \in (C, 1)$, it is taken as understood that $C < 1$. The notation $c_n \sim d_n$ means that $C_1 \leqslant c_n/d_n \leqslant C_2$ for the relevant range of n. Similar notation is used for sequences of functions. $\langle x \rangle$ denotes the greatest integer $\leqslant x$. In the sequel, we assume that $w = e^{-Q} \in \mathscr{E}$.

2. TECHNICAL LEMMAS

We begin by recalling some technical results from the first part of this paper [16]. Throughout we assume that $w \in \mathscr{E}$. Recall that Φ_t is defined by (1.9).

LEMMA 2.1. (a) *For* $u \geqslant C$,

$$Q'(a_u) \sim u T(a_u)^{1/2}. \tag{2.1}$$

Given fixed $\beta > 0$, *we have for large* u

$$T(a_{\beta u}) \sim T(a_u). \tag{2.2}$$

Given fixed $\alpha > 1$,

$$\frac{a_{\alpha u}}{a_u} - 1 \sim \frac{1}{T(a_u)}. \tag{2.3}$$

For some $\delta > 0$, $C_3 > 0$,

$$T(a_u) \leqslant C_3 u^{2(1-\delta)}. \tag{2.4}$$

(b) *There exists* C_1, *such that for* $0<s\leqslant C_1$, *and* $s/2\leqslant t\leqslant s$,

$$\Phi_s(x)\sim\Phi_t(x), \qquad x\in[-1,1]. \tag{2.5}$$

Moreover, for $0<s<t\leqslant C$,

$$\sup_{x\in[-1,1]}\frac{\Phi_s(x)}{\Phi_t(x)}\leqslant C_2\sqrt{\log\left(2+\frac{t}{s}\right)}. \tag{2.6}$$

(c) *Let* $L>0$. *For* $t\in(0,t_0)$, $|x|,|y|\leqslant a_{1/t}$ *such that*

$$|x-y|\leqslant Lt\Phi_t(x), \tag{2.7}$$

we have

$$w(x)\sim w(y); \qquad \Phi_t(x)\sim\Phi_t(y). \tag{2.8}$$

Proof. (a) This is part of Lemma 2.2 in [16].

(b) This is Lemmas 3.1(b) and 7.1(b) in [16].

(c) This is Lemma 3.2(b) in [16], with a trivial modification to the proof there. ∎

Next, we present a restricted range inequality involving a suitable modification of $\Phi_{1/n}$:

LEMMA 2.2. *Let* $0<p\leqslant\infty$, $s>1$. *For* $n\geqslant 1$, *define*

$$\Psi_n(x):=\left(1-\left[\frac{x}{a_n}\right]^2\right)^2+T(a_n)^{-2}. \tag{2.9}$$

Then for some $n\geqslant 1$, $0\leqslant l\leqslant n$ *and* $P\in\mathscr{P}_n$,

$$\|Pw\Psi_n^{l/4}\|_{L_p(-1,1)}\leqslant C_1\,\|Pw\Psi_n^{l/4}\|_{L_p(-a_{s(l+n)},\,a_{s(l+n)})}. \tag{2.10}$$

Moreover,

$$\|Pw\Psi_n^{l/4}\|_{L_p(1\geqslant|x|\geqslant a_{sn})}\leqslant C_1e^{-C_2nT(a_n)^{-1/2}}\,\|Pw\Psi_n^{l/4}\|_{L_p(-a_{s(l+n)},\,a_{s(l+n)})}. \tag{2.11}$$

Here $C_j\neq C_j(n,P_n,l)$, $j=1,2$. *The same result holds for a fixed* l (*with constants depending on* l) *if we replace* $\Psi_n^{1/4}$ *by* $\Phi_{1/n}$.

Remark. Note that (2.4) shows that for some $C_3>0$, and large enough n,

$$nT(a_n)^{-1/2}\geqslant n^{C_3}. \tag{2.12}$$

Proof. For $l=0$, this is Lemma 2.3 in [16]. Next note that

$$T(a_n)^{-2} \leqslant \Psi_n(x) \leqslant C \qquad \text{in } [-1, 1].$$

Write $l=4j+k$, $0 \leqslant k \leqslant 3$. Then as $P\Psi_n^j$ is a polynomial of degree at most $n+l \leqslant 2n$, the case $l=0$ of (2.11) gives

$$\begin{aligned} \|Pw\Psi_n^{l/4}\|_{L_p(1 \geqslant |x| \geqslant a_{s(l+n)})} &= \|Pw\Psi_n^j \Psi_n^{k/4}\|_{L_p(1 \geqslant |x| \geqslant a_{s(l+n)})} \\ &\leqslant C_1 \|Pw\Psi_n^j\|_{L_p(1 \geqslant |x| \geqslant a_{s(l+n)})} \\ &\leqslant C_1 e^{-C_2 n T(a_n)^{-1/2}} \|Pw\Psi_n^j\|_{L_p(-a_{s(l+n)}, a_{s(l+n)})}. \end{aligned}$$

Our lower bound for Ψ_n allows us to continue this as

$$\leqslant C_1 T(a_n)^{k/2} e^{-C_2 n T(a_n)^{-1/2}} \|Pw\Psi_n^{j+k/4}\|_{L_p(-a_{s(l+n)}, a_{s(l+n)})}$$

and since (2.12) holds, we deduce (2.11) and hence (2.10). Since $\Psi_n^{1/4} \sim \Phi_{1/n}$ uniformly in $[-1, 1]$ and in n, we also obtain (2.10), (2.11) for a fixed l, with $\Psi_n^{1/4}$ replaced by $\Phi_{1/n}$. ∎

We shall need an extension of the Markov–Bernstein inequality (1.16):

Lemma 2.3. *Let* $0 < p \leqslant \infty$ *and* Ψ_n *be defined by* (2.9) *for* $n \geqslant 1$. *Then for* $n \geqslant 1$, $0 \leqslant l \leqslant n$, $P_n \in \mathscr{P}_n$,

$$\|P_n^{(l+1)} w \Psi_n^{(l+1)/4}\|_{L_p[-1, 1]} \leqslant C_1 \{n + l T(a_n)^{1/2}\} \|P_n^{(l)} w \Psi_n^{l/4}\|_{L_p[-1, 1]} \tag{2.13}$$

$$\leqslant C_2 n(l+1) \|P_n^{(l)} w \Psi_n^{l/4}\|_{L_p[-1, 1]}, \tag{2.14}$$

where $C_j \neq C_j(n, l, P_n)$, $j=1, 2$. *The same result holds for a fixed* l *(with constants depending on* l*) if we replace* $\Psi_n^{1/4}$ *by* $\Phi_{1/n}$.

Proof. It suffices to prove the result for large n. We first construct suitable polynomial approximations of $\Psi_n^{1/4}$. To do this, we use the Christoffel functions $\lambda_m(u, x)$ associated with the ultraspherical weight

$$u(x) := (1-x^2)^{-3/4}, \qquad x \in (-1, 1).$$

It is known [24, p. 36] that $\lambda_m(u, x)^{-1}$ is a polynomial of degree $2m-2$ such that uniformly in m and $x \in (-1, 1)$,

$$\lambda_m(u, x)^{-1} \sim m[1 - x^2 + m^{-2}]^{1/4} \tag{2.15}$$

and given $A > 0$, for $|x| \leqslant 1 - Am^{-2}$,

$$|\lambda_m'(u, x)/\lambda_m(u, x)| \leqslant C[1 - x^2 + m^{-2}]^{-1}. \tag{2.16}$$

We choose $m := m(n)$ to be the greatest integer $\leqslant T(a_n)^{1/2}$ and set

$$R_n(x) := \left[\frac{1}{m}\lambda_m^{-1}\left(u, \frac{x}{a_{3n}}\right)\right]^2.$$

Then R_n has degree $o(n)$ by (2.4). Moreover, (2.15) and (2.16) give

$$R_n(x) \sim \Phi_{1/n}(x) \sim \Psi_n(x)^{1/4}, \qquad x \in [-a_{3n}, a_{3n}] \tag{2.17}$$

and

$$|R_n'(x)/R_n(x)| \leqslant C\Psi_n(x)^{-1/2}, \qquad x \in [-a_{3n}, a_{3n}]. \tag{2.18}$$

Next, write $l = 4j + k$, $0 \leqslant k \leqslant 3$. Let $Q_1 := P_n^{(l)}$. Let $1 < s < \frac{3}{2}$. From Lemma 2.2,

$$\begin{aligned}
\|P_n^{(l+1)} w \Psi_n^{(l+1)/4}\|_{L_p[-1,1]} &\leqslant C_1 \|Q_1' w \Psi_n^{(l+1)/4}\|_{L_p[-a_{s(l+1+n)}, a_{s(l+1+n)}]} \\
&\leqslant C_2 \|Q_1' \Psi_n^j R_n^k w \Psi_n^{1/4}\|_{L_p[-a_{3n}, a_{3n}]} \\
&\leqslant C_3\{\|(Q_1 \Psi_n^j R_n^k)' w \Psi_n^{1/4}\|_{L_p[-a_{3n}, a_{3n}]} \\
&\quad + j\|Q_1 \Psi_n^{j-1} \Psi_n' R_n^k w \Psi_n^{1/4}\|_{L_p[-a_{3n}, a_{3n}]} \\
&\quad + k\|Q_1 \Psi_n^j R_n^{k-1} R_n' w \Psi_n^{1/4}\|_{L_p[-a_{3n}, a_{3n}]}\} \\
&=: C_3(T_1 + T_2 + T_3).
\end{aligned} \tag{2.19}$$

Here by (1.16), applied to $Q_1 \Psi_n^j R_n^k$, which has degree at most $n + l + o(n) \leqslant \frac{5}{2}n$, and as

$$\Psi_n^{1/4} \sim \Phi_{1/(3n)} \qquad \text{in } [-1, 1]$$

we have

$$\begin{aligned}
T_1 &\leqslant C_4 n \|Q_1 \Psi_n^j R_n^k w\|_{L_p[-1,1]} \\
&\leqslant C_5 n \|Q_1 \Psi_n^j R_n^k w\|_{L_p[-a_{3n}, a_{3n}]} \\
&\leqslant C_6 n \|P_n^{(l)} w \Psi_n^{l/4}\|_{L_p[-a_{3n}, a_{3n}]}
\end{aligned}$$

by first (2.10) and then (2.17). Next,

$$|\Psi_n'(x)| = 4\left|1 - \left(\frac{x}{a_n}\right)^2\right| \frac{|x|}{a_n^2} \leqslant \frac{4}{a_n^2}\Psi_n(x)^{1/2}.$$

Using (2.17), we see that

$$T_2 \leqslant C_7 l \, \|P_n^{(l)} w \Psi_n^{l/4-1/4}\|_{L_p[-a_{3n},\, a_{3n}]} \leqslant C_8 l T(a_n)^{1/2} \, \|P_n^{(l)} w \Psi_n^{l/4}\|_{L_p[-a_{3n},\, a_{3n}]}$$

as $\Psi_n^{-1/4}(x) \leqslant T(a_n)^{1/2}$. Similarly, using (2.18) and then (2.17),

$$\begin{aligned} T_3 &\leqslant C_9 \, \|Q_1 \Psi_n^j R_n^k \Psi_n^{-1/2} w \Psi_n^{1/4}\|_{L_p[-a_{s(l+n)},\, a_{s(l+n)}]} \\ &\leqslant C_{10} \, \|P_n^{(l)} w \Psi_n^{l/4-1/4}\|_{L_p[-a_{3n},\, a_{3n}]} \end{aligned}$$

as before. So T_3 admits the same estimate as T_2. Substituting into (2.19) gives (2.13) and then (2.14) follows as $T(a_n)^{1/2} = O(n)$. ∎

Finally, we present an estimate of differences:

LEMMA 2.4. *Let* $0 < \delta < 1$; $L, M > 0$; $0 < p \leqslant \infty$.

(a) *Let* $s \in (0, 1]$ *and* $[a, b]$ *be contained in one of the ranges*

$$|x| \leqslant a_{1/t} \left(1 - \left[\frac{s}{2\delta a_{1/t}}\right]^2\right) \tag{2.20}$$

or

$$1 > |x| \geqslant a_{1/t} \left(1 + \left[\frac{s}{2\delta a_{1/t}}\right]^2\right). \tag{2.21}$$

Then

$$\int_a^b |f(x \pm s\Phi_t(x))| \, dx \leqslant 2(1-\delta)^{-1} \int_{\bar{a}}^{\bar{b}} |f(x)| \, dx, \tag{2.22}$$

where

$$\begin{Bmatrix} \bar{a} \\ \bar{b} \end{Bmatrix} := \begin{Bmatrix} \inf \\ \sup \end{Bmatrix} \{x \pm s\Phi_t(x) : x \in [a, b]\}. \tag{2.23}$$

(b) *Let* $r \geqslant 1$, $t \in (0, 1/M)$, $h \in (0, Mt)$, *and* $[a, b]$ *be contained in one of the ranges* (2.20), (2.21) *with* $s = Mrt$. *Define* $\bar{a}$ *and* $\bar{b}$ *by* (2.23) *with* $s = Mrt$. *Assume, moreover, that*

$$[a, b] \subset [-a_{L/t}, a_{L/t}]. \tag{2.24}$$

Then

$$\|w(x)\,\Delta^r_{h\Phi_t(x)}(g, x, (-1, 1))\|_{L_p[a,b]}$$
$$\leqslant C \inf_{P \in \mathscr{P}_{r-1}} \|w(g-P)\|_{L_p[\bar{a},\bar{b}]} \leqslant C\,\|wg\|_{L_p[\bar{a},\bar{b}]}. \tag{2.25}$$

Here $C \neq C(a, b, t, h, g)$.

Proof. (a) Define $\sigma = \pm 1$ and $u(x) := x + \sigma s\Phi_t(x)$. We shall assume that $[a, b]$ is contained in the range (2.20) and also $a \geqslant 0$. The case where $a < 0$ is similar, as is the case when $[a, b]$ is contained in the range (2.21). Then for $x \in [a, b]$,

$$u'(x) = 1 + \frac{\sigma s}{2\sqrt{1-(x/a_{1/t})}}\left(-\frac{1}{a_{1/t}}\right) \geqslant 1 - \delta$$

by (2.20). Hence u is increasing in $[a, b]$, and writing $v := u(x)$,

$$\int_a^b |f(x \pm s\Phi_t(x))|\,dx$$
$$= \int_a^b |f(u(x))|\,dx$$
$$\leqslant \frac{1}{1-\delta}\int_a^b |f(u(x))|\,u'(x)\,dx = (1-\delta)^{-1}\int_{u(a)}^{u(b)} |f(v)|\,dv.$$

So we have (2.22). The extra 2 in (2.22) takes care of having to split $[a, b]$ into two intervals if $a < 0 < b$.

(b) We shall assume that $p < \infty$. The proof is easier for $p = \infty$. Let us assume that $[a, b]$ is contained in the range (2.20) with $s = Mrt$. Now

$$w(x)\,\Delta^r_{h\Phi_t(x)}(g, x, (-1, 1)) = \sum_{i=0}^{r} \binom{r}{i} (-1)^i\, w(x)\, g\left(x + \left(\frac{r}{2} - i\right) h\Phi_t(x)\right).$$

Next, (2.8) shows that

$$w(x) \sim w\left(x \pm \left(\frac{r}{2} - i\right) h\Phi_t(x)\right)$$

uniformly in i and for $|x| \leqslant a_{L/t}$ and $h \leqslant Mt$. (It is only here that we need (2.24).) Then using (a), we obtain

$$\|w(x)\, \Delta^r_{h\Phi_t(x)}(g, x, (-1, 1))\|^p_{L_p[a, b]} \leqslant C \sup_{0 \leqslant i \leqslant r} \int_a^b |gw|^p \left(x \pm \frac{ih}{2}\, \Phi_t(x) \right) dx$$

$$\leqslant \frac{2C}{1-\delta} \int_{\bar{a}}^{\bar{b}} |gw|^p\,(x)\, dx.$$

Note that for $0 \leqslant i \leqslant r$, and $h \leqslant Mt$, our hypothesis (2.20) with $s = Mrt$ gives

$$|x| \leqslant a_{1/t} \left(1 - \left(\frac{Mrt}{2\delta a_{1/t}} \right)^2 \right) \leqslant a_{1/t} \left(1 - \left(\frac{ih}{2\delta a_{1/t}} \right)^2 \right)$$

so the range restrictions of (a) are satisfied. Finally note that for $P \in \mathscr{P}_{r-1}$,

$$\Delta^r_{h\Phi_t(x)}(P, x, (-1, 1)) \equiv 0.$$

Hence

$$\|w(x)\, \Delta^r_{h\Phi_t(x)}(g, x, (-1, 1))\|_{L_p[a, b]}$$

$$= \|w(x)\, \Delta^r_{h\Phi_t(x)}(g - P, x, (-1, 1))\|_{L_p[a, b]} \leqslant C\, \|w(g-P)\|_{L_p[\bar{a}, \bar{b}]}.$$

Now take inf's over P. ∎

3. A CRUCIAL INEQUALITY

In this section, we establish a crucial inequality using ideas of Z. Ditzian [8].

THEOREM 3.1. *Let* $w \in \mathscr{E}$. *Let* $r \geqslant 1$, $L > 0$, $0 < p \leqslant \infty$, $P_n \in \mathscr{P}_n$. *If* $0 < p < 1$, *assume also the Markov–Bernstein inequality* (1.16). *Define* $P \in \mathscr{P}_{r-1}$ *by*

$$P(x) := P_n(x) - \int_{a_{Ln}}^{x} \int_{a_{Ln}}^{u_{r-1}} \cdots \int_{a_{Ln}}^{u_1} P_n^{(r)}(u_0)\, du_0\, du_1 \cdots du_{r-1}. \tag{3.1}$$

Then for some $C_j \neq C_j(n, P)$, $j = 1, 2$,

$$\|(P_n - P)\, w\|_{L_p[a_{Ln}, 1]} \leqslant C_1 (nT(a_n)^{1/2})^{-r}\, \|P_n^{(r)} w\|_{L_p[-1, 1]} \tag{3.2}$$

$$\leqslant C_2 n^{-r}\, \|P_n^{(r)} w \Phi^r_{1/n}\|_{L_p[-1, 1]}. \tag{3.3}$$

Our method of proof follows that of [8, 3]. The chief difficulty lies in the case $p < 1$. We first deal with $p \geqslant 1$, following the approach of [8].

LEMMA 3.2. *Assume the hypotheses of Theorem* 3.1 *with* $p \geqslant 1$. *Then* $\forall g \in L_p[a_{Ln}, 1]$,

$$\left\| w(x) \int_{a_{Ln}}^{x} g(u)\, du \right\|_{L_p[a_{Ln}, 1]} \leqslant C_1 (nT(a_n)^{1/2})^{-1} \, \|gw\|_{L_p[a_{Ln}, 1]}. \tag{3.4}$$

Here $C_1 \neq C_1(n, g)$.

Proof. This is similar to Lemma 11.4.1 in [9, p. 187], to Lemma 6.2 in [8], and to Lemma 4.2 in [3], but we provide the details. We begin by noting that for $x \geqslant t > 0$,

$$w(x)^{1/2} \int_{t}^{x} w(u)^{-1/2} \, Q'(u)\, du = 2\left(1 - \left(\frac{w(x)}{w(t)}\right)^{1/2}\right) \leqslant 2. \tag{3.5}$$

Next for $n \geqslant n_0$ and $u \geqslant a_{Ln}$,

$$Q'(u) \geqslant C_1 \, Q'(a_{Ln}) \geqslant C_2 nT(a_n)^{1/2}. \tag{3.6}$$

(See (2.1).) Consequently for $x \geqslant a_{Ln}$,

$$w(x) \left| \int_{a_{Ln}}^{x} g(u)\, du \right| \leqslant [C_2 nT(a_n)^{1/2}]^{-1} \, w(x)^{1/2} \int_{a_{Ln}}^{x} |gw| \times (u)\, Q'(u)\, w^{-1/2}(u)\, du. \tag{3.7}$$

We shall need a consequence of Jensen's inequality for integrals, applied with the power function t^p, $p \geqslant 1$: For non-negative measures μ and μ-measurable functions f,

$$\left| \int f\, d\mu \right|^p \leqslant \left(\int |f|^p \, d\mu \right) \left(\int d\mu \right)^{p-1}. \tag{3.8}$$

Now we turn to the proof of (3.4):

$p = \infty$: Here (3.7) implies that for $x \geqslant a_{Ln}$,

$$\begin{aligned} w(x) \left| \int_{a_{Ln}}^{x} g(u)\, du \right| &\leqslant [C_2 nT(a_n)^{1/2}]^{-1} \, \|gw\|_{L_\infty[a_{Ln}, 1]} \\ &\quad \times w(x)^{1/2} \int_{a_{Ln}}^{x} Q'(u)\, w^{-1/2}(u)\, du \\ &\leqslant 2[C_2 nT(a_n)^{1/2}]^{-1} \, \|gw\|_{L_\infty[a_{Ln}, 1]}. \end{aligned}$$

Here we have used (3.5). So we obtain (3.4).

$p<\infty$: Here applying (3.7),

$$\begin{aligned}
&\left\| w(x) \int_{a_{Ln}}^{x} g(u)\, du \right\|_{L_p[a_{Ln},1]} \\
&\quad \leqslant [C_2 n T(a_n)^{1/2}]^{-1} \left[\int_{a_{Ln}}^{1} \left[w(x)^{1/2} \int_{a_{Ln}}^{x} |gw| \right.\right. \\
&\qquad \left.\left. \times (u)\, Q'(u)\, w^{-1/2}(u)\, du \right]^{p} dx \right]^{1/p} \\
&\quad \leqslant [C_2 n T(a_n)^{1/2}]^{-1} \left[\int_{a_{Ln}}^{1} 2^{p-1} w(x)^{1/2} \int_{a_{Ln}}^{x} |gw|^{p} \right. \\
&\qquad \left. \times (u)\, Q'(u)\, w^{-1/2}(u)\, du\, dx \right]^{1/p}
\end{aligned}$$

by (3.8) with $d\mu(u) := w(x)^{1/2}\, Q'(u)\, w(u)^{-1/2}\, du$ on $[a_{Ln}, x]$ and as (3.5) shows that $\int d\mu \leqslant 2$. Now

$$\begin{aligned}
&\int_{a_{Ln}}^{1} w(x)^{1/2} \int_{a_{Ln}}^{x} |gw|^{p}(u)\, Q'(u)\, w(u)^{-1/2}\, du\, dx \\
&\quad = \int_{a_{Ln}}^{1} |gw|^{p}(u) \left[\int_{u}^{1} w(x)^{1/2}\, Q'(u)\, dx \right] w(u)^{-1/2}\, du \\
&\quad \leqslant C_3 \int_{a_{Ln}}^{1} |gw|^{p}(u) \left[\int_{u}^{1} w(x)^{1/2}\, Q'(x)\, dx \right] w(u)^{-1/2}\, du \\
&\quad = 2C_3 \, \|gw\|^{p}_{L_p[a_{Ln},1]}.
\end{aligned}$$

In the second last line, we used the quasi-monotonicity of Q'. So (3.4) follows. ∎

Proof of (3.3) *for* $p \geqslant 1$. This follows by induction on n from Lemma 3.2: Firstly for $r=1$, Lemma 3.2 applied to $g = P'_n$ gives

$$\left\| w(x) \int_{a_{Ln}}^{x} P'_n(u_0)\, du_0 \right\|_{L_p[a_{Ln},1]} \leqslant C_1 (nT(a_n)^{1/2})^{-1} \, \|P'_n w\|_{L_p[a_{Ln},1]}$$

and then the identity (3.1) gives (3.2). As $\Phi_{1/n}(x) \geqslant T(a_n)^{-1/2}$, (3.3) also follows. Next for $r=2$, Lemma 3.2 applied to

$$g(u_1) := \int_{a_{Ln}}^{u_1} P''_n(u_0)\, du_0$$

gives

$$\begin{aligned}
&\left\| w(x) \int_{a_{Ln}}^{x} \int_{a_{Ln}}^{u_n} P_n''(u_0)\, du_0\, du_1 \right\|_{L_p[a_{Ln},1]} \\
&\quad = \left\| w(x) \int_{a_{Ln}}^{x} g(u_1)\, du_1 \right\|_{L_p[a_{Ln},1]} \\
&\quad \leqslant C_2 (nT(a_n)^{1/2})^{-1} \, \|gw\|_{L_p[a_{Ln},1]} \\
&\quad = C_2 (nT(a_n)^{1/2})^{-1} \left\| w(x) \int_{a_{Ln}}^{x} P_n''(u_0)\, du_0 \right\|_{L_p[a_{Ln},1]} \\
&\quad \leqslant C_3 (nT(a_n)^{1/2})^{-2} \, \|wP_n''\|_{L_p[a_{Ln},1]},
\end{aligned}$$

by Lemma 3.2 again. Then (3.1) gives the result. Clearly after applying Lemma 3.2 r times to the right-hand side of (3.1), we get the result. ∎

We break down the proof of Theorem 3.1 for $p<1$ into a number of lemmas.

LEMMA 3.3. *Let $0<p<1$, $n \geqslant r \geqslant 1$, $L>0$. Let $P_n \in \mathscr{P}_n$, $S \in \mathscr{P}_{r-1}$ and let*

$$g(x) := (P_n - S)(x). \tag{3.9}$$

Let

$$I_n(x) := \left\| \, |g'w|^{1-p}(u) \left(\frac{w(x)}{w(u)} \right)^{1/2} \right\|^{p/(1-p)}_{L_\infty[a_{Ln},x]}, \qquad x \geqslant a_{Ln}. \tag{3.10}$$

Then

$$\begin{aligned}
\int_{a_{Ln}}^{1} I_n(x)\, dx \leqslant C \Bigg[\sum_{j=1}^{r-1} & (nT(a_n)^{1/2})^{-(j-1)p} \, \|w(P_n - S)^{(j)}\|^p_{L_p[a_{Ln},1]} \\
& + (nT(a_n)^{1/2})^{-(r-1)p} \, \|wP_n^{(r)}\|^p_{L_p[-1,1]} \Bigg].
\end{aligned} \tag{3.11}$$

Here C is independent of n, P_n, S.

Proof. Note first that

$$I_n(x) = \left\| \, |g'w|^{p}(u) \left(\frac{w(x)}{w(u)} \right)^{p/(2(1-p))} \right\|_{L_\infty[a_{Ln},x]}.$$

Let $\tau := \delta(nT(a_n)^{1/2})^{-1}$, where $\delta > 0$ is so small that for $n \geqslant 1$, $R \in \mathscr{P}_n$,

$$T(a_n)^{-1/2} \|R'w\|_{L_p[-1,1]} \leqslant \|R'w\Phi_{1/n}\|_{L_p[-1,1]} \leqslant \frac{n}{2\delta} \|Rw\|_{L_p[-1,1]}. \quad (3.12)$$

Given $x \geqslant a_{Ln}$, we define

$$k_0 := k_0(x) := \max\{k\colon x-(k+1)\,\tau \geqslant a_{Ln}\}.$$

We see that

$$I_n(x) \leqslant \max_{0 \leqslant k \leqslant k_0(x)} \left\| |g'w|^p(u) \left(\frac{w(x)}{w(u)}\right)^{p/(2(1-p))} \right\|_{L_\infty[x-(k+1)\tau,\, x-k\tau]} + \left\| |g'w|^p(u) \left(\frac{w(x)}{w(u)}\right)^{p/(2(1-p))} \right\|_{L_\infty[a_{Ln},\, x-(k_0+1)\tau]}.$$

Now for $u \in [x-(k+1)\,\tau, x-k\tau]$, (recall that $x-(k+1)\,\tau \geqslant a_{Ln}$)

$$\frac{w(x)}{w(u)} \leqslant e^{Q(x-k\tau)-Q(x)}$$

and by (3.6),

$$Q(x)-Q(x-k\tau) \geqslant C_2 nT(a_n)^{1/2}\, k\tau = C_2\delta k. \quad (3.13)$$

Thus

$$\left(\frac{w(x)}{w(u)}\right)^{p/(2(1-p))} \leqslant q^k, \qquad u \in [x-(k+1)\,\tau, x-k\tau],$$

where $q \in (0, 1)$ is independent of x, u, k. Then

$$I_n(x) \leqslant \max_{0 \leqslant k \leqslant k_0(x)} q^k \|g'w\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]} + q^{k_0(x)} \|g'w\|^p_{L_\infty[a_{Ln},\, x-(k_0+1)\tau]}$$

$$\leqslant \sum_{k=0}^{k_0(x)} q^k \|g'w\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]} + q^{k_0(x)} \|g'w\|^p_{L_\infty[a_{Ln},\, x-(k_0+1)\tau]}.$$

Let us set $I_n(x) := 0 =: (g'w)(x)$, $x > 1$, so that the previous inequality remains valid for $x \geqslant 1$. This device simplifies the subsequent argument. Then

$$\int_{a_{Ln}}^{1} I_n(x)\,dx = \sum_{m=0}^{\infty} \int_{a_{Ln}+m\tau}^{a_{Ln}+(m+1)\tau} I_n(x)\,dx$$
$$\leqslant \sum_{m=0}^{\infty} \int_{a_{Ln}+m\tau}^{a_{Ln}+(m+1)\tau} \left[\sum_{k=0}^{k_0(x)} q^k \,\|g'w\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]} \right.$$
$$\left. + q^{k_0(x)} \,\|g'w\|^p_{L_\infty[a_{Ln},\, x-(k_0+1)\tau]} \right] dx.$$

We observe that

$$\int_{a_{Ln}+m\tau}^{a_{Ln}+(m+1)\tau} \|g'w\|^p_{L_\infty[x-(k+1)\tau,\, x-k\tau]}\, dx = \int_{a_{Ln}+(m-k-1)\tau}^{a_{Ln}+(m-k)\tau} \|g'w\|^p_{L_\infty[x,\, x+\tau]}\, dx.$$

Recall that if $x \in [a_{Ln}+m\tau, a_{Ln}+(m+1)\tau]$, then $m \geqslant k_0 \geqslant m-1$, so

$$\int_{a_{Ln}}^{1} I_n(x)\,dx$$
$$\leqslant \sum_{m=0}^{\infty} \left[\sum_{k=0}^{m-1} \int_{a_{Ln}+(m-k-1)\tau}^{a_{Ln}+(m-k)\tau} q^k \,\|g'w\|^p_{L_\infty[x,\, x+\tau]}\, dx \right.$$
$$\left. + 2q^{m-1} \int_{a_{Ln}}^{a_{Ln}+\tau} \|g'w\|^p_{L_\infty[a_{Ln},\, x]}\, dx \right]$$
$$\leqslant \sum_{s=0}^{\infty} \int_{a_{Ln}+s\tau}^{a_{Ln}+(s+1)\tau} \|g'w\|^p_{L_\infty[x,\, x+\tau]}\, dx \sum_{\substack{(m,k)\\ s=m-k-1}} q^k$$
$$+ 2 \int_{a_{Ln}}^{a_{Ln}+\tau} \|g'w\|^p_{L_\infty[a_{Ln},\, x]}\, dx \,\frac{1}{q(1-q)}$$
$$\leqslant C_3 \left[\sum_{s=0}^{\infty} \int_{a_{Ln}+s\tau}^{a_{Ln}+(s+1)\tau} \|g'w\|^p_{L_\infty[x,\, x+\tau]}\, dx + \int_{a_{Ln}}^{a_{Ln}+\tau} \|g'w\|^p_{L_\infty[a_{Ln},\, x]}\, dx \right]. \tag{3.14}$$

Now for $u \in [x, x+\tau]$ (recall that $p < 1$)

$$|g'(u)|^p = \left| \sum_{j=1}^{n} \frac{g^{(j)}(x)}{(j-1)!} (u-x)^{j-1} \right|^p$$
$$\leqslant \sum_{j=1}^{n} |g^{(j)}(x)|^p \,\tau^{(j-1)p}$$
$$= \sum_{j=1}^{r-1} |(P_n - S)^{(j)}(x)|^p \,\tau^{(j-1)p} + \sum_{j=r}^{n} |P_n^{(j)}(x)|^p \,\tau^{(j-1)p}.$$

Since also $w(u) \leqslant w(x)$ for $u \in [x, x+\tau]$, we obtain (recall our convention $g'(x) = 0,\ x > 1$)

$$\sum_{s=0}^{\infty} \int_{a_{Ln}+s\tau}^{a_{Ln}+(s+1)\tau} \|g'w\|^p_{L_\infty[x, x+\tau]}\, dx$$

$$\leqslant \sum_{j=1}^{r-1} \tau^{(j-1)p} \|(P_n - S)^{(j)} w\|^p_{L_p[a_{Ln}, 1]}$$

$$+ \tau^{(r-1)p} \sum_{j=r}^{n} \tau^{(j-r)p} \|P_n^{(j)} w\|^p_{L_p[a_{Ln}, 1]}$$

$$\leqslant \sum_{j=1}^{r-1} \tau^{(j-1)p} \|(P_n - S)^{(j)} w\|^p_{L_p[a_{Ln}, 1]}$$

$$+ \tau^{(r-1)p} \|P_n^{(r)} w\|^p_{L_p[-1, 1]} \sum_{j=r}^{n} \left(\frac{\tau}{2\delta} nT(a_n)^{1/2}\right)^{(j-r)p}$$

(by (3.12))

$$\leqslant \sum_{j=1}^{r-1} \left(\frac{\delta}{nT(a_n)^{1/2}}\right)^{(j-1)p} \|(P_n - S)^{(j)} w\|^p_{L_p[a_{Ln}, 1]}$$

$$+ (1 - 2^{-p})^{-1} \left(\frac{\delta}{nT(a_n)^{1/2}}\right)^{(r-1)p} \|P_n^{(r)} w\|^p_{L_p[-1, 1]}. \qquad (3.15)$$

Here we have used our choice $\tau = \delta(nT(a_n)^{1/2})^{-1}$. In much the same way, we can use Taylor series to estimate the second term in the right-hand side of (3.14). Together (3.14) and (3.15) then give the result. ∎

LEMMA 3.4. *Let $0 < p < 1$, $n \geqslant r \geqslant 1$, $L > 0$. Let $P_n \in \mathscr{P}_n$, $S \in \mathscr{P}_{r-1}$ and let*

$$(P_n - S)(a_{Ln}) = 0. \qquad (3.16)$$

Then for some C independent of n, P_n, S

$$\|w(P_n - S)\|_{L_p[a_{Ln}, 1]} \leqslant C(nT(a_n)^{1/2})^{-1} \|w(P_n' - S')\|^p_{L_p[a_{Ln}, 1]}$$

$$\times \left[\sum_{j=1}^{r-1} (nT(a_n)^{1/2})^{-(j-1)(1-p)} \|w(P_n - S)^{(j)}\|^{1-p}_{L_p[a_{Ln}, 1]} \right.$$

$$\left. + (nT(a_n)^{1/2})^{-(r-1)(1-p)} \|wP_n^{(r)}\|^{1-p}_{L_p[-1, 1]} \right]. \qquad (3.17)$$

Proof. Let g be given by (3.9). Note that $g(a_{Ln})=0$ by our hypothesis (3.16). In particular

$$g(x)=\int_{a_{Ln}}^{x} g'(u)\,du.$$

Then

$$\begin{aligned}\Delta &:= \|w(P_n-S)\|_{L_p[a_{Ln},1]}=\|wg\|_{L_p[a_{Ln},1]}\\ &=\left\{\int_{a_{Ln}}^{1}\left|\int_{a_{Ln}}^{x}(g'w)(u)\,\frac{w(x)}{w(u)}\,du\right|^{p}dx\right\}^{1/p}\\ &\leqslant\left\{\int_{a_{Ln}}^{1}\left\||g'w|^{1-p}(u)\left(\frac{w(x)}{w(u)}\right)^{1/2}\right\|^{p}_{L_\infty[a_{Ln},x]}\right.\\ &\qquad\times\left.\left\{\int_{a_{Ln}}^{x}|g'w|^{p}(u)\left(\frac{w(x)}{w(u)}\right)^{1/2}du\right\}^{p}dx\right\}^{1/p}.\end{aligned}$$

Now we apply Hölder's inequality with parameters $r=1/(1-p)$, $\sigma=1/p$, so that $1/\sigma+1/r=1$:

$$\begin{aligned}\Delta\leqslant&\left\{\int_{a_{Ln}}^{1}\left\||g'w|^{1-p}(u)\left(\frac{w(x)}{w(u)}\right)^{1/2}\right\|^{p/(1-p)}_{L_\infty[a_{Ln},x]}dx\right\}^{(1-p)/p}\\ &\times\left\{\int_{a_{Ln}}^{1}\int_{a_{Ln}}^{x}|g'w|^{p}(u)\left(\frac{w(x)}{w(u)}\right)^{1/2}du\,dx\right\}=:T_1\times T_2.\end{aligned}\tag{3.18}$$

We see from (3.10), (3.11) of Lemma 3.3 that

$$\begin{aligned}T_1&=\left\{\int_{a_{Ln}}^{1}I_n(x)\,dx\right\}^{(1-p)/p}\\ &\leqslant C_1\left[\sum_{j=1}^{r-1}(nT(a_n)^{1/2})^{-(j-1)(1-p)}\|w(P_n-S)^{(j)}\|^{1-p}_{L_p[a_{Ln},1]}\right.\\ &\qquad\left.+(nT(a_n)^{1/2})^{-(r-1)(1-p)}\|wP_n^{(r)}\|^{1-p}_{L_p[-1,1]}\right].\end{aligned}\tag{3.19}$$

Of course C_1 is not the same C as in (3.11), but is independent of n and P_n. Next, using (3.6),

$$T_2=\int_{a_{Ln}}^1 |g'w|^p(u)\int_u^1\left(\frac{w(x)}{w(u)}\right)^{1/2}dx\,du$$

$$\leqslant C_3(nT(a_n)^{1/2})^{-1}\int_{a_{Ln}}^1 |g'w|^p(u)\left[w(u)^{-1/2}\int_u^1 w(x)^{1/2}\,Q'(x)\,dx\right]du$$

$$=2C_3(nT(a_n)^{1/2})^{-1}\int_{a_{Ln}}^1 |g'w|^p(u)\,du$$

$$=2C_3(nT(a_n)^{1/2})^{-1}\,\|w(P_n'-S')\|^p_{L_p[a_{Ln},1]}.$$

Combining this with (3.18) and (3.19), we obtain the result. ∎

Proof of Theorem 3.1 *for* $p<1$. Let $P_n\in\mathscr{P}_n$ and let the corresponding P be given by (3.1). We begin by noting an extension of (3.17): For $0\leqslant l<r$,

$$\begin{aligned}\|w(P_n-P)^{(l)}\|_{L_p[a_{Ln},1]}&\\ \leqslant C(nT(a_n)^{1/2})^{-1}&\,\|w(P_n-P)^{(l+1)}\|^p_{L_p[a_{Ln},1]}\\ \times\Bigg[\sum_{j=l+1}^{r-1}&(nT(a_n)^{1/2})^{-(j-l-1)(1-p)}\,\|w(P_n-P)^{(j)}\|^{1-p}_{L_p[a_{Ln},1]}\\ +(nT(a_n)^{1/2})&^{-(r-l-1)(1-p)}\,\|wP_n^{(r)}\|^{1-p}_{L_p[-1,1]}\Bigg].\end{aligned}\tag{3.20}$$

The case $l=0$ of (3.20) is just (3.17) with $S=P$. The case $l>0$ follows by applying Lemma 3.4 to the polynomial $(P_n-P)^{(l)}\in\mathscr{P}_{n-l}$ and with r in (3.17) replaced by $r-l$. (Note that $(P_n-P)^{(l)}(a_{Ln})=0$, $0\leqslant l<r$, so the hypothesis (3.16) is fulfilled). We use (3.20) and backward induction to show that

$$\|w(P_n-P)^{(k)}\|_{L_p[a_{Ln},1]}\leqslant C(nT(a_n)^{1/2})^{k-r}\,\|wP_n^{(r)}\|_{L_p[-1,1]},\tag{3.21}$$

$k=r-1, r-2, ..., 0$. Of course Theorem 3.1 is the case $k=0$ of (3.21).

$\boldsymbol{k=r-1}$. Here (3.20) with $l=r-1$ gives

$$\begin{aligned}\|w(P_n-P)^{(r-1)}\|_{L_p[a_{Ln},1]}&\\ \leqslant C(nT(a_n)^{1/2})^{-1}&\,\|w(P_n-P)^{(r)}\|^p_{L_p[a_{Ln},1]}\,\|wP_n^{(r)}\|^{1-p}_{L_p[-1,1]}\\ =C(nT(a_n)^{1/2})^{-1}&\,\|wP_n^{(r)}\|_{L_p[-1,1]}.\end{aligned}$$

Assume (3.21) true for $\boldsymbol{r-1, r-2, ..., k+1}$. We prove (3.21) for $l=k$ by substituting (3.21) for $l=r-1, r-2, ..., k+1$ into (3.20) with $l=k$. We obtain

$$\begin{aligned}
&\|w(P_n-P)^{(k)}\|_{L_p[a_{Ln},1]} \\
&\quad\leqslant C(nT(a_n)^{1/2})^{-1}\,[(nT(a_n)^{1/2})^{k+1-r}\,\|wP_n^{(r)}\|_{L_p[-1,1]}]^p \\
&\qquad\times\left[\sum_{j=k+1}^{r-1}(nT(a_n)^{1/2})^{-(j-k-1)(1-p)+(j-r)(1-p)}\,\|wP_n^{(r)}\|_{L_p[-1,1]}^{1-p}\right. \\
&\qquad\left.+(nT(a_n)^{1/2})^{-(r-k-1)(1-p)}\,\|wP_n^{(r)}\|_{L_p[-1,1]}^{1-p}\right] \\
&\quad\leqslant C_1(nT(a_n)^{1/2})^{k-r}\,\|wP_n^{(r)}\|_{L_p[-1,1]}.
\end{aligned}$$

So we have (3.21) for k. ∎

4. PROOF OF THEOREM 1.3

In this section, we shall prove Theorem 1.3. We first prove two lemmas, which together give most of the proof. Recall first that for $t>0$,

$$\bar{K}_{r,p}(f,w,t^r):=\inf_{\deg(P)\leqslant 1/t}\{\|(f-P)\,w\|_{L_p[-1,1]}+t^r\,\|P^{(r)}w\Phi_t^r\|_{L_p[-1,1]}\};$$

$$\begin{aligned}
\omega_{r,p}(f,w,t)=&\sup_{0<h\leqslant t}\|w\Delta^r_{h\Phi_t(x)}(f,x,(-1,1))\|_{L_p(|x|\leqslant a_{1/(2t)})} \\
&+\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|_{L_p(1\geqslant|x|\geqslant a_{1/(4t)})};
\end{aligned}$$

and

$$\begin{aligned}
\bar{\omega}_{r,p}(f,w,t)=&\left[\frac{1}{t}\int_0^t\|w\Delta^r_{h\Phi_t(x)}(f,x,(-1,1))\|^p_{L_p(|x|\leqslant a_{1/(2t)})}\,dh\right]^{1/p} \\
&+\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|_{L_p(1\geqslant|x|\geqslant a_{1/(4t)})}.
\end{aligned}$$

Throughout this section, we set $q:=\min\{1,p\}$ and we assume the hypotheses of Theorem 1.3 (unless otherwise specified). We begin by estimating $\omega_{r,p}$ above in terms of $\bar{K}_{r,p}$.

LEMMA 4.1. *Let $L>0$, $0<p\leqslant\infty$, $r\geqslant 1$. Then for $t\in(0,C_1)$ we have*

$$\omega_{r,p}(f,w,Lt)\leqslant C_2\bar{K}_{r,p}(f,w,t^r), \tag{4.1}$$

where $C_j\neq C_j(f,t)$, $j=1,2$.

Proof. Fix $t<1$ and determine P_n of degree at most $n:=\langle 1/t\rangle\leqslant 1/t$ such that

$$\|(f-P_n)\,w\|_{L_p[-1,1]}+t^r\,\|P_n^{(r)}w\Phi_t^r\|_{L_p[-1,1]}\leqslant 2\bar{K}_{r,p}(f,w,t^r). \tag{4.2}$$

We shall show that for $0<h\leqslant Lt$,

$$\|w\Delta^r_{h\Phi_{Lt}(x)}(f,x,(-1,1))\|_{L_p(|x|\leqslant a_{1/(2Lt)})}\leqslant C_1\bar{K}_{r,p}(f,w,t^r) \tag{4.3}$$

and

$$\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|_{L_p(1\geqslant|x|\geqslant a_{1/(4Lt)})}\leqslant C_2\bar{K}_{r,p}(f,w,t^r), \tag{4.4}$$

which by definition of $\omega_{r,p}$ implies (4.1).

Proof of (4.3). Now

$$\begin{aligned}
&\|w\Delta^r_{h\Phi_{Lt}(x)}(f,x,(-1,1))\|^q_{L_p(|x|\leqslant a_{1/(2Lt)})}\\
&\quad\leqslant\|w\Delta^r_{h\Phi_{Lt}(x)}(f-P_n,x,(-1,1))\|^q_{L_p(|x|\leqslant a_{1/(2Lt)})}\\
&\qquad+\|w\Delta^r_{h\Phi_{Lt}(x)}(P_n,x,(-1,1))\|^q_{L_p(|x|\leqslant a_{1/(2Lt)})}\\
&\quad\leqslant C_3\,\|(f-P_n)\,w\|^q_{L_p[-1,1]}+\|w\Delta^r_{h\Phi_{Lt}(x)}(P_n,x,(-1,1))\|^q_{L_p(|x|\leqslant a_{1/(2Lt)})},
\end{aligned} \tag{4.5}$$

by Lemma 2.4(b). Note that given any $A>0$, (2.3), (2.2) and then (2.4) show that for small enough t and $|x|\leqslant a_{1/(2Lt)}$

$$1-\frac{|x|}{a_{1/(Lt)}}\geqslant 1-\frac{a_{1/(2Lt)}}{a_{1/(Lt)}}\geqslant\frac{C}{T(a_{1/t})}\geqslant At^2$$

so x lies in the range (2.20) with $s=Lr\cdot Lt$ and Lemma 2.4(b) is applicable. We now proceed to the (fairly complicated) estimation of the second term on the right-hand side of (4.5). Recall that $\Delta^r_h S\equiv 0$ for $S\in\mathscr{P}_{r-1}$. In particular this applies to

$$S(u)=\sum_{l=0}^{r-1}\frac{P_n^{(l)}(x)}{l!}\,(u-x)^l.$$

Then using Taylor series and this remark, we see that

$$\Delta^r_{h\Phi_{Lt}(x)} P_n(x) = \sum_{k=0}^{r} \binom{r}{k} (-1)^k P_n\left(x + \left(\frac{r}{2} - k\right) h\Phi_{Lt}(x)\right)$$

$$= \sum_{k=0}^{r} \binom{r}{k} (-1)^k \sum_{l=r}^{n} \frac{((r/2-k)\, h\Phi_{Lt}(x))^l}{l!} P_n^{(l)}(x). \quad (4.6)$$

Then we deduce that

$$\|w\Delta^r_{h\Phi_{Lt}(x)}(P_n, x, (-1, 1))\|^q_{L_p[-1,1]}$$
$$\leqslant \sum_{k=0}^{r} \binom{r}{k}^q \sum_{l=r}^{n} \frac{([r/2]\, h)^{lq}}{l!^q} \|P_n^{(l)} w\Phi^l_{Lt}\|^q_{L_p[-1,1]}. \quad (4.7)$$

Now as $n = \langle 1/t \rangle$, we have by (2.5) that

$$\Phi_{1/n}(x) \sim \Phi_{Lt}(x) \sim \Phi_t(x) \sim \Psi_n^{1/4}(x), \qquad x \in (-1, 1), \quad (4.8)$$

where the constants in $\sim$ are independent of x, t, and $n = n(t) = \langle 1/t \rangle$. Recall that Ψ_n was defined by (2.9). By repeated application of our Markov–Bernstein inequality Lemma 2.3,

$$\|P_n^{(l)} w\Phi^l_{Lt}\|_{L_p[-1,1]} \leqslant C_1^l \|P_n^{(l)} w\Psi_n^{l/4}\|_{L_p[-1,1]}$$
$$\leqslant C_1^l \|P_n^{(r)} w\Psi_n^{r/4}\|_{L_p[-1,1]} C_2^{l-r} \prod_{j=r}^{l-1} (n + jT(a_n)^{1/2}).$$

Since $T(a_n)^{1/2} = o(n)$, we see that given $\varepsilon > 0$, we have for $n \geqslant n_0(\varepsilon)$, $r \leqslant l \leqslant n$,

$$\prod_{j=r}^{l-1} (n + jT(a_n)^{1/2}) \leqslant C_3 n^{l-r} [\varepsilon^{l-r} l! + 1]$$

where C_3 and n_0 do not depend on l, n (nor on h, L above). It is also important that C_1, C_2 above are independent of ε. We deduce that for $n \geqslant n_0(\varepsilon)$, (or equivalently for small t), and $0 < h \leqslant Lt$,

$$\|w\Delta^r_{h\Phi_{Lt}(x)}(P_n, x, (-1, 1))\|^q_{L_p[-1,1]}$$
$$\leqslant 2^r C_4 \|P_n^{(r)} w\Psi_n^{r/4}\|^q_{L_p[-1,1]} \left(\frac{r}{2} h\right)^{rq} \sum_{l=r}^{n} \frac{([r/2]\, hC_1 C_2 n)^{(l-r)q}}{l!^q} [\varepsilon^{(l-r)q} l!^q + 1$$
$$\leqslant C_5 \|P_n^{(r)} w\Phi_t^r\|^q_{L_p[-1,1]} t^{rq} \sum_{l=r}^{\infty} \left(\frac{r}{2} LC_1 C_2\right)^{(l-r)q} \left[\varepsilon^{(l-r)q} + \frac{1}{l!^q}\right]$$
$$\leqslant C_6 t^{rq} \|P_n^{(r)} w\Phi_t^r\|^q_{L_p[-1,1]}$$

as $nt \leqslant 1$ and provided ε is chosen so small that $[r/2]\, LC_1 C_2 \varepsilon < 1$. This is possible as ε did not depend on L, n, C_1, C_2 but can be made arbitrarily small for $n \geqslant n_0$. Then using (4.2) and (4.5), we obtain (4.3).

Proof of (4.4). Next, we recall Lemma 3.1 from [9], proved there for weights on the whole real line, but valid without change for $[-1, 1]$: For $\xi > 0$,

$$\inf_{S \in \mathscr{P}_{r-1}} \|(f-S)\, w\|_{L_p(1 \geqslant |x| \geqslant \xi)}$$
$$\leqslant 2^{4/q-3}\Big[\inf_{S \in \mathscr{P}_{r-1}} \|(f-S)\, w\|_{L_p[\xi, 1]} + \inf_{S \in \mathscr{P}_{r-1}} \|(f-S)\, w\|_{L_p[-1, -\xi]}\Big].$$

We shall apply this with $\xi = a_{1/(4Lt)}$ and use Theorem 3.1 to estimate $\inf_{S \in \mathscr{P}_{r-1}} \|(f-S)\, w\|_{L_p[\xi, 1]}$. The term on $[-1, -\xi]$ is handled similarly. For P_n determined by (4.2), choose $P \in \mathscr{P}_{r-1}$ by the identity (3.1) with L replaced by $1/(4L)$. Then as $a_{1/(4Lt)} \geqslant a_{n/(4L)}$,

$$\inf_{S \in \mathscr{P}_{r-1}} \|(f-S)\, w\|^q_{L_p[a_{1/(4Lt)}, 1]}$$
$$\leqslant \|(f-P)\, w\|^q_{L_p[a_{1/(4Lt)}, 1]}$$
$$\leqslant \|(f-P_n)\, w\|^q_{L_p[a_{1/(4Lt)}, 1]} + \|(P_n - P)\, w\|^q_{L_p[a_{n/(4L)}, 1]}$$
$$\leqslant [2\bar{K}_{r, p}(f, w, t^r)]^q + C_5 (nT(a_n)^{1/2})^{-rq}\, \|P_n^{(r)} w\|^q_{L_p[-1, 1]}$$

(by (4.2) and (3.2))

$$\leqslant [2\bar{K}_{r, p}(f, w, t^r)]^q + C_5 t^{rq}\, \|P_n^{(r)} w \Phi_t^r\|^q_{L_p[-1, 1]} \leqslant C_6 \bar{K}_{r, p}(f, w, t^r)^q.$$

Here we have used $\frac{1}{2} \leqslant nt \leqslant 1$, (4.8), and then (4.2). ∎

The converse direction is more difficult. We first prove:

LEMMA 4.2. *There exist* C_j, $j = 1, 2$, *and* $0 < \varepsilon_0 < 1$ *such that if* $0 < \lambda < \varepsilon_0$, *and* $0 < s, t \leqslant C_1$ *with*

$$\lambda \leqslant \frac{s}{t} \leqslant \varepsilon_0 \tag{4.9}$$

we have

$$\bar{\omega}_{r, p}(f, w, s) \leqslant C_2 \bar{\omega}_{r, p}(f, w, t). \tag{4.10}$$

Here C_j, $j = 1, 2$ *and* ε_0 *do not depend on* f, s, t (*but depend on* λ).

Proof. We do this for $p<\infty$; $p=\infty$ is much easier. We split

$$\begin{aligned}\bar{\omega}_{r,p}(f,w,s)^p \leqslant \frac{2^p}{s}\int_0^s [\,&\|w\Delta^r_{h\Phi_s(x)}(f,x,(-1,1))\|^p_{L_p(|x|\leqslant a_{1/(3t)})}\\ &+\|w\Delta^r_{h\Phi_s(x)}(f,x,(-1,1))\|^p_{L_p(a_{1/(3t)}\leqslant|x|\leqslant a_{1/(2s)})}]\,dh\\ &+2^p\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|^p_{L_p(a_{1/(4s)}\leqslant|x|\leqslant 1)}\\ =:{}& \mathscr{T}_1+\mathscr{T}_2+\mathscr{T}_3. \qquad (4.11)\end{aligned}$$

First, as $a_{1/(4s)}\geqslant a_{1/(4t)}$,

$$\mathscr{T}_3\leqslant 2^p\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|^p_{L_p(a_{1/(4t)}\leqslant|x|\leqslant 1)}\leqslant 2^p\bar{\omega}^p_{r,p}(f,w,t). \qquad (4.12)$$

Next, by Lemma 2.4(b),

$$\mathscr{T}_2\leqslant 2^p\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|^p_{L_p(a_{1/(4t)}\leqslant|x|\leqslant 1)}\leqslant 2^p\bar{\omega}^p_{r,p}(f,w,t). \qquad (4.13)$$

In applying that lemma, we note that our range of integration is of the form (2.20). Moreover, in working out $\bar{a}$, we used the fact that $\Phi_s(x)$ is a decreasing function of $x\in[0,a_{1/(2s)}]$, so

$$\begin{aligned}&\inf\{x-Mrs\Phi_s(x)\colon a_{1/(3t)}\leqslant x\leqslant a_{1/(2s)}\}\\ &\quad=a_{1/(3t)}-Mrs\Phi_s(a_{1/(3t)})\\ &\quad\geqslant a_{1/(3t)}-MrCt\Phi_t(a_{1/(3t)})\end{aligned}$$

(by (2.6))

$$\geqslant a_{1/(3t)}+o(1/T(a_{1/t}))\geqslant a_{1/(4t)}$$

for small t, see (2.3) and (2.4). Note that the bound on how small t should be in no way depends on s or λ. It is more difficult to handle $\mathscr{T}_1$. Let us divide $J:=[-a_{1/(3t)},a_{1/(3t)}]$ into $O(1/s)$ intervals J_k such that

$$|J_k|\leqslant s\Phi_s(x), \qquad x\in J_k$$

for all k. Here $|J_k|$ denotes the length of J_k. Formally we may do this by choosing a large positive integer $n\sim 1/s$ and a partition

$$-a_{1/3t}=\tau_0<\tau_1<\tau_2<\cdots<\tau_n=a_{1/3t}$$

with

$$\int_{\tau_k}^{\tau_{k+1}} \Phi_s^{-1}(x)\,dx \Big/ \int_{\tau_0}^{\tau_n} \Phi_s^{-1}(x)\,dx = \frac{1}{n}, \qquad 0 \leqslant k \leqslant n$$

and we set $J_k := [\tau_k, \tau_{k+1}]$. Note that then by (2.8),

$$\Phi_s(x) \sim \Phi_s(y);\ w(x) \sim w(y), \qquad x, y \in J_k \tag{4.14}$$

with constants independent of x, y, s, k. Then

$$\begin{aligned}\mathscr{T}_1 &= \frac{2^p}{s} \int_0^s \sum_k \int_{J_k} |w \Delta^r_{h\Phi_s(x)}(f, x, (-1, 1))|^p\,dx\,dh \\ &\leqslant C \sum_k w^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^s |\Delta^r_{h\Phi_s(x)}(f, x, (-1, 1))|^p\,dh\,dx \\ &= C \sum_k w^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^{s\Phi_s(x)/\Phi_t(x)} |\Delta^r_{u\Phi_t(x)}(f, x, (-1, 1))|^p \frac{\Phi_t(x)}{\Phi_s(x)}\,du\,dx.\end{aligned}$$

Now from (2.6), for some $C \neq C(s, t, \lambda)$,

$$\sup_{x \in (-1, 1)} \frac{s\Phi_s(x)}{t\Phi_t(x)} \leqslant C \frac{s}{t} \sqrt{\log\left(2 + \frac{s}{t}\right)} \leqslant 1$$

if $s/t \leqslant \varepsilon_0$. Here $\varepsilon_0 \neq \varepsilon_0(s, t, \lambda)$. Moreover, by (2.5), for $\lambda \leqslant s/t \leqslant \varepsilon_0$,

$$C_3 \leqslant \frac{\Phi_s(x)}{\Phi_t(x)} \leqslant C_4, \qquad x \in (-1, 1).$$

Then

$$\begin{aligned}\mathscr{T}_1 &\leqslant C_5 \sum_k w^p(\tau_k) \int_{J_k} \frac{1}{s} \int_0^t |\Delta^r_{u\Phi_t(x)}(f, x, (-1, 1))|^p\,du\,dx \\ &\leqslant C_6 \frac{1}{t} \int_0^t \int_J |w \Delta^r_{u\Phi_t(x)}(f, x, (-1, 1))|^p\,dx\,du \\ &\leqslant C_6 \bar{\omega}_{r,p}(f, w, t)^p.\end{aligned}$$

In the last line we used (4.14). Together with (4.12) and (4.13), this gives the result. ∎

LEMMA 4.3. *Let $0 < p \leqslant \infty$, $r \geqslant 1$. There exist $C_1, C_2, C_3 > 0$ independent of f and t such that for $0 < t < C_1$,*

$$\bar{K}_{r,p}(f, w, t^r) \leqslant C_2 \bar{\omega}_{r,p}(f, w, C_3 t). \tag{4.15}$$

Proof. Let $t>0$ and $n:=\langle 1/t\rangle$. By Theorem 1.2, for $n\geqslant C_3$ we can choose $P_n\in\mathscr{P}_n$ such that

$$\|(f-P_n)\,w\|_{L_p[-1,1]}\leqslant C_7 \inf_{\rho\in[3/4,1]} \bar{\omega}_{r,p}\left(f,w,C_8\frac{\rho}{n}\right)\leqslant C_7\bar{\omega}_{r,p}(f,w,C_8 t) \tag{4.16}$$

for $t<C_1<\frac{1}{4}$, since then we can choose

$$\rho:=\rho(t):=nt=\langle 1/t\rangle\, t\in[\tfrac{3}{4},1].$$

We shall show that

$$t^r\,\|P_n^{(r)}w\Phi_t^r\|_{L_p[-1,1]}\leqslant C_9\bar{\omega}_{r,p}(f,w,C_8 t). \tag{4.17}$$

Once we have done this it follows that

$$\begin{aligned}\bar{K}_{r,p}(f,w,t^r)&\leqslant\|(f-P_n)\,w\|_{L_p[-1,1]}+t^r\,\|P_n^{(r)}w\Phi_t^r\|_{L_p[-1,1]}\\&\leqslant C_{10}\bar{\omega}_{r,p}(f,w,C_8 t).\end{aligned}$$

To prove (4.17), we set $s:=\delta t$, where $\delta\leqslant\frac{1}{8}$ so that

$$\frac{1}{2s}\geqslant\frac{4}{t}\geqslant 2n,\qquad t\leqslant t_0 \tag{4.18}$$

and δ is also so small that we can apply Lemma 4.2 (with t replaced by $C_8 t$) to deduce that

$$\bar{\omega}_{r,p}(f,w,s)\leqslant C_{11}\bar{\omega}_{r,p}(f,w,C_8 t). \tag{4.19}$$

Of course $\delta\neq\delta(f,t)$ but C_{11} depends on δ. We proceed to prove (4.17). Note first that applying (4.6) to the monomial x^r (with Lt replaced by s) gives

$$r!(h\Phi_s(x))^r=\Delta^r_{h\Phi_s(x)}x^r=\sum_{k=0}^{r}\binom{r}{k}(-1)^k\frac{(([r/2]-k)\,h\Phi_s(x))^r}{r!}\,r!$$

so we can rewrite (4.6) as

$$\begin{aligned}&\Delta^r_{h\Phi_s(x)}P_n(x)-(h\Phi_s(x))^r\,P_n^{(r)}(x)\\&\quad=\sum_{k=0}^{r}\binom{r}{k}(-1)^k\sum_{l=r+1}^{n}\frac{(([r/2]-k)\,h\Phi_s(x))^l}{l!}\,P_n^{(l)}(x).\end{aligned}$$

Moreover, as $s\sim t\sim 1/n$, we have (see (2.5))

$$\Phi_s(x)\sim\Phi_t(x)\sim\Phi_{1/n}(x)\sim\Psi_n^{1/4}(x),\qquad x\in(-1,1) \tag{4.20}$$

with the constants in $\sim$ independent of n and t. Then

$$\begin{aligned}\Gamma &:= \|w[\Delta^r_{h\Phi_s(x)} P_n(x) - (h\Phi_s(x))^r P_n^{(r)}(x)]\|^q_{L_p[-a_{1/(2s)}, a_{1/(2s)}]} \\ &\leqslant \sum_{k=0}^r \binom{r}{k}^q \sum_{l=r+1}^n \frac{([r/2]\, h)^{lq}}{l!^q} \|P_n^{(l)} w\Phi_s^l\|^q_{L_p[-1,1]} \\ &\leqslant \sum_{k=0}^r \binom{r}{k}^q \sum_{l=r+1}^n \frac{(C_{12}[r/2]\, h)^{lq}}{l!^q} \|P_n^{(l)} w\Psi_n^{l/4}\|^q_{L_p[-1,1]} \\ &\leqslant C_{14} \|P_n^{(r)} w\Psi_n^{r/4}\|^q_{L_p[-1,1]} \left(C_{12}\frac{r}{2}h\right)^{rq} \sum_{l=r+1}^n \frac{(C_{12}C_{13}n[r/2]\, h)^{(l-r)q}}{l!^q} l!^q\end{aligned}$$

by repeated applications of our Markov–Bernstein inequality (2.14). It is important here that C_{12}, C_{13} are independent of t, n, h, P_n, l. Now if $nh \leqslant \Delta$, where Δ is a fixed positive small enough number, Lemma 2.2 and (4.18) above allow us to continue this as

$$\Gamma \leqslant \tfrac{1}{2} h^{rq} \|P_n^{(r)} w\Phi_s^r\|^q_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}.$$

It is crucial that Δ is independent of t, h, n, P_n, l. We deduce that for $0 < h \leqslant \Delta/n$ and hence for $0 < h \leqslant \Delta t$, and hence for $0 < h \leqslant \Delta s$,

$$\begin{aligned}&\|w\Delta^r_{h\Phi_s(x)} P_n(x)\|^q_{L_p[-a_{1/(2s)}, a_{1/(2s)}]} \\ &\quad\geqslant h^{rq} \|wP_n^{(r)}\Phi_s^r\|^q_{L_p[-a_{1/(2s)}, a_{1/(2s)}]} \\ &\qquad - \|w[\Delta^r_{h\Phi_s(x)} P_n(x) - (h\Phi_s(x))^r P_n^{(r)}(x)]\|^q_{L_p[-a_{1/(2s)}, a_{1/(2s)}]} \\ &\quad\geqslant \tfrac{1}{2} h^{rq} \|wP_n^{(r)}\Phi_s^r\|^q_{L_p[-a_{1/(2s)}, a_{1/(2s)}]} \geqslant C_{14} h^{rq} \|wP_n^{(r)}\Phi_s^r\|^q_{L_p[-1,1]}.\end{aligned}$$

(In the last step, we again used $1/2s \geqslant 2n$ and Lemma 2.2.) Raising to the (p/q)th power, integrating for h from 0 to Δs, and using (4.20), we obtain

$$\frac{1}{\Delta s}\int_0^{\Delta s} \|w\Delta^r_{h\Phi_s(x)} P_n(x)\|^p_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}\, dh \geqslant C_{15} s^{rp} \|P_n^{(r)} w\Phi_t^r\|^p_{L_p[-1,1]}.$$

Assuming, as we can, that $\Delta < 1$, we obtain

$$\begin{aligned}&t^{rp} \|P_n^{(r)} w\Phi_t^r\|^p_{L_p[-1,1]} \\ &\quad\leqslant \frac{C_{16}}{s}\int_0^s \|w\Delta^r_{h\Phi_s(x)} P_n(x)\|^p_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}\, dh \\ &\quad\leqslant \frac{C_{17}}{s}\int_0^s \{\|w\Delta^r_{h\Phi_s(x)}(P_n - f)(x)\|^p_{L_p[-a_{1/(2s)}, a_{1/(2s)}]} \\ &\qquad + \|w\Delta^r_{h\Phi_s(x)} f(x)\|^p_{L_p[-a_{1/(2s)}, a_{1/(2s)}]}\}\, dh \\ &\quad\leqslant C_{18}\{\|w(P_n - f)\|^p_{L_p[-1,1]} + \bar{\omega}_{r,p}(f, w, s)^p\}\end{aligned}$$

(by (2.25))

$$\leqslant C_{19}\bar{\omega}_{r,p}(f, w, C_8 t)^p$$

by (4.16) and (4.19). So we have (4.17). ∎

We can now turn to the

Proof of Theorem 1.3. From Lemmas 4.1 and 4.3, for any fixed $L > 0$, $0 < t < t_0$, we have

$$\bar{\omega}_{r,p}(f, w, Lt) \leqslant \omega_{r,p}(f, w, Lt) \leqslant C_1 \bar{K}_{r,p}(f, w, t^r)$$
$$\leqslant C_2 \bar{\omega}_{r,p}(f, w, C_3 t) \leqslant C_2 \omega_{r,p}(f, w, C_3 t). \qquad (4.21)$$

Here it is important that C_3 is independent of L, f, t. Fix $M > 0$ and choose $L = MC_3$ and set $s = C_3 t$ to deduce that

$$\omega_{r,p}(f, w, Ms) \leqslant C_2 \omega_{r,p}(f, w, s).$$

So we have (1.19). Similarly (4.21) gives

$$\bar{\omega}_{r,p}(f, w, Ms) \leqslant C_2 \bar{\omega}_{r,p}(f, w, s).$$

Then (4.21) gives

$$\omega_{r,p}(f, w, s) \sim \bar{\omega}_{r,p}(f, w, s) \sim \bar{K}_{r,p}(f, w, s^r)$$

with constants in $\sim$ independent of f, s. ∎

5. PROOF OF COROLLARIES 1.4, 1.5, and 1.6

First, Corollary 1.4 follows from (1.19) of Theorem 1.3 and Theorem 1.2. We turn to the

Proof of Corollary 1.5. Let $t \leqslant C < \frac{1}{2}$ and choose $n := \langle 1/t \rangle$ and $l := \langle \log_2(1/t) \rangle$, so that

$$\frac{1}{2t} \leqslant 2^l \leqslant \frac{1}{t}$$

and

$$\tfrac{1}{2} \leqslant n/2^l \leqslant 2.$$

For $j \geqslant 1$, let P_j^* be a best polynomial approximation of degree $\leqslant j$ to f, so that

$$\|(f - P_j^*)\, w\|_{L_p[-1,1]} = E_j[f]_{w,p}.$$

Moreover, we set $P_{2^{-1}}^* := P_0^*$. Then by (1.19) of Theorem 1.3,

$$\begin{aligned}\omega_{r,p}(f, w, t)^q &\leqslant C_1 \omega_{r,p}(f, w, n^{-1})^q \leqslant C_2 \bar{K}_{r,p}(f, w, n^{-r})^q \\ &\leqslant C_3 \inf_{\deg(P) \leqslant n} \{\|(f-P)\, w\|_{L_p[-1,1]}^q \\ &\quad + n^{-rq} \|P^{(r)} w \Phi_{1/n}^r\|_{L_p[-1,1]}^q\} \\ &\leqslant C_4 \{\|(f - P_{2^{l-1}}^*)\, w\|_{L_p[-1,1]}^q + n^{-rq} \|P_{2^{l-1}}^{*(r)} w \Phi_{2^{-l}}^r\|_{L_p[-1,1]}^q\}\end{aligned}$$

(by (2.5))

$$\leqslant C_5 \{E_{2^{l-1}}[f]_{w,p}^q + t^{rq} \sum_{k=0}^{l-1} \|\{P_{2^k}^* - P_{2^{k-1}}^*\}^{(r)}\, w \Phi_{2^{-l}}^r\|_{L_p[-1,1]}^q\}$$

(as $P_{2^{-1}}^{*(r)} \equiv 0$)

$$\leqslant C_6 \{E_{2^{l-1}}[f]_{w,p}^q + t^{rq} \sum_{k=0}^{l-1} (\log 2^{l-k})^{rq/2} \|\{P_{2^k}^* - P_{2^{k-1}}^*\}^{(r)}\, w \Phi_{2^{-k}}^r\|_{L_p[-1,1]}^q$$

(by (2.6))

$$\leqslant C_7 \{E_{2^{l-1}}[f]_{w,p}^q + t^{rq} \sum_{k=0}^{l-1} (l-k)^{rq/2}\, 2^{rkq} \|\{P_{2^k}^* - P_{2^{k-1}}^*\}\, w\|_{L_p[-1,1]}^q\}$$

(by (2.14): recall r is fixed)

$$\leqslant C_8 \{E_{2^{l-1}}[f]_{w,p}^q + t^{rq} \sum_{k=0}^{l-1} (l-k)^{rq/2}\, 2^{rkq} E_{2^{k-1}}[f]_{w,p}^q\}$$

$$\leqslant C_9 t^{rq} \sum_{k=-1}^{l} (l-k+1)^{rq/2}\, 2^{rkq} E_{2^k}[f]_{w,p}^q. \quad \blacksquare$$

We turn to the

Proof of Corollary 1.6(a). We shall separately show that for $0 < h \leqslant t$,

$$\|w \Delta_{h\Phi_t(x)}^r (f, x, (-1, 1))\|_{L_p(|x| \leqslant a_{1/(2t)})} \leqslant C t^r \|f^{(r)} w \Phi_t^r\|_{L_p[-1,1]} \tag{5.1}$$

and

$$\inf_{P \in \mathscr{P}_{r-1}} \|(f-P)\, w\|_{L_p(1 \geqslant |x| \geqslant a_{1/(4t)})} \leqslant C t^r \|f^{(r)} w \Phi_t^r\|_{L_p[-1,1]}. \tag{5.2}$$

The corollary follows immediately from these two inequalities. We do these for $1 \leqslant p < \infty$; the case $p = \infty$ is easier.

Proof of (5.1). First note that for $u > 0$,

$$\Delta_u^r(f, x, (-1, 1))$$
$$= \left| \int_{-u/2}^{u/2} \int_{-u/2}^{u/2} \cdots \int_{-u/2}^{u/2} f^{(r)}(x + t_1 + t_2 + \cdots + t_r)\, dt_1\, dt_2 \cdots dt_r \right|$$
$$\leqslant u^{r-1} \int_{-ru/2}^{ru/2} |f^{(r)}(x+s)|\, ds. \tag{5.3}$$

Now we claim that

$$\Phi_t(x+s) \sim \Phi_t(x); \; w(x+s) \sim w(x) \tag{5.4}$$

uniformly for

$$|x| \leqslant a_{1/(2t)}; \; |s| \leqslant \frac{rt}{2}\, \Phi_t(x).$$

This follows from Lemma 2.1(c) provided that for this range of x, s

$$|x+s| \leqslant a_{1/t}. \tag{5.5}$$

Now if $x \geqslant 0$,

$$|x+s| \leqslant x_0 := x + \frac{rt}{2}\left[\sqrt{1 - \frac{x}{a_{1/t}}} + T(a_{1/t})^{-1/2}\right].$$

Here by (2.3), (2.4),

$$\frac{rt}{2\sqrt{1-(x/a_{1/t})}} \leqslant \frac{rt}{2\sqrt{1-(a_{1/(2t)}/a_{1/t})}} \leqslant CtT(a_{1/t})^{1/2} = o(1), \qquad t \to 0+.$$

So, by definition of x_0 and this last inequality,

$$1 - \frac{x_0}{a_{1/t}} \geqslant 1 - \frac{x}{a_{1/t}} - o(1)\left(1 - \frac{x}{a_{1/t}}\right) - \frac{rt}{2a_{1/t}}\, T(a_{1/t})^{-1/2}$$
$$\geqslant \left(1 - \frac{a_{1/(2t)}}{a_{1/t}}\right)(1 - o(1)) - o(T(a_{1/t})^{-1}) \geqslant CT(a_{1/t})^{-1} > 0$$

by (2.4), (2.3). So we have (5.5) for small enough t. Then from that and then (5.3) and then (5.4),

$$\begin{aligned}|w(x)\,\Delta^r_{h\Phi_t(x)}(f,x,(-1,1))| &\leqslant C_4 w(x)(h\Phi_t(x))^{r-1}\int_{-rh\Phi_t(x)/2}^{rh\Phi_t(x)/2}|f^{(r)}(x+s)|\,ds\\ &\leqslant C_5\frac{h^r}{h\Phi_t(x)}\int_{-rh\Phi_t(x)/2}^{rh\Phi_t(x)/2}|w\Phi_t^r f^{(r)}|\,(x+s)\,ds. \qquad (5.6)\end{aligned}$$

Now for $p>1$, the maximal function operator

$$M[g](x):=\sup_{u>0}\frac{1}{2u}\int_{-u}^{u}|g(x+s)|\,ds$$

is bounded from L_p to L_p, so

$$\begin{aligned}\|w\Delta^r_{h\Phi_t(x)}(f,x,(-1,1))\|_{L_p(|x|\leqslant a_{1/(2t)})} &\leqslant C_6 h^r\,\|M[w\Phi_t^r f^{(r)}]\|_{L_p[-1,1]}\\ &\leqslant C_7 h^r\,\|w\Phi_t^r f^{(r)}\|_{L_p[-1,1]},\end{aligned}$$

that is, we have (5.1). In the case where $p=1$, we integrate (5.6), interchange integrals, and obtain (5.1) again, using also (2.8).

Proof of (5.2). Write $n:=\langle 1/t\rangle$. Note that $1/(4t)\geqslant n/4$ so that

$$\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|_{L_p(1\geqslant|x|\geqslant a_{1/(4t)})}\leqslant\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|_{L_p(1\geqslant|x|\geqslant a_{n/4})}.$$

In fact it suffices to estimate

$$\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|_{L_p[a_{n/4},1]}$$

as a similar estimate holds for the range $[-1,-a_{n/4}]$; recall again from [8, Lemma 3.1] that

$$\begin{aligned}&\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|_{L_p[a_{n/4}\leqslant|x|\leqslant 1]}\\ &\quad\leqslant 2^{4/q-3}\Big[\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|_{L_p[a_{n/4},1]}+\inf_{P\in\mathscr{P}_{r-1}}\|(f-P)\,w\|_{L_p[-1,-a_{n/4}]}\Big].\end{aligned}$$

Now if $r=1$, we can use Lemma 3.2 to deduce that (recall that $p \geqslant 1$ and that Lemma 3.2 did not require a Markov–Bernstein inequality)

$$\begin{aligned}
\inf_{P \in \mathscr{P}_0} &\|(f-P)\,w\|_{L_p[a_{n/4},\,1]} \\
&\leqslant \|(f(x)-f(a_{n/4}))\,w(x)\|_{L_p[a_{n/4},\,1]} \\
&= \left\| w(x) \int_{a_{n/4}}^{x} f'(u)\,du \right\|_{L_p[a_{n/4},\,1]} \leqslant C_1[nT(a_n)]^{-1/2}\, \|f'w\|_{L_p[a_{n/4},\,1]} \\
&\leqslant C_2 n^{-1}\, \|f'w\Phi_t\|_{L_p[a_{n/4},\,1]}.
\end{aligned}$$

Next, as in induction hypothesis, assume that for $k=1, 2, ..., r-1$, we have

$$\inf_{P \in \mathscr{P}_{k-1}} \|(g-P)\,w\|_{L_p[a_{n/4},\,1]} \leqslant C_3 n^{-k}\, \|g^{(k)} w\Phi_t^k\|_{L_p[a_{n/4},\,1]}, \tag{5.7}$$

where $C \neq C(g, n)$. (We have just proved this for $k=1$.) Applying this with $k=r-1$ to $g := f'$, we can choose $S \in \mathscr{P}_{r-2}$ such that

$$\|(f'-S)\,w\|_{L_p[a_{n/4},\,1]} \leqslant C_3 n^{-(r-1)}\, \|f^{(r)} w\Phi_t^{r-1}\|_{L_p[a_{n/4},\,1]}.$$

Set

$$P_1(x) := f(a_{n/4}) + \int_{a_{n/4}}^{x} S(u)\,du.$$

Then

$$\begin{aligned}
\inf_{P \in \mathscr{P}_{r-1}} \|(f-P)\,w\|_{L_p[a_{n/4},\,1]} &\leqslant \|(f-P_1)\,w\|_{L_p[a_{n/4},\,1]} \\
&\leqslant \left\| w(x) \int_{a_{n/4}}^{x} (f'-S)(u)\,du \right\|_{L_p[a_{n/4},\,1]} \\
&\leqslant C_4[nT(a_n)]^{-1/2}\, \|(f'-S)\,w\|_{L_p[a_{n/4},\,1]}
\end{aligned}$$

(by Lemma 3.2)

$$\begin{aligned}
&\leqslant C_5 n^{-r} T(a_n)^{-1/2}\, \|f^{(r)} w\Phi_t^{r-1}\|_{L_p[a_{n/4},\,1]} \\
&\leqslant C_5 n^{-r}\, \|f^{(r)} w\Phi_t^{r}\|_{L_p[a_{n/4},\,1]}.
\end{aligned}$$

So we have (5.7) for r and hence also (5.2). ∎

In proving Corollary 1.6(b), we need

LEMMA 5.1. *Let $L > 0$, $0 < p \leqslant \infty$. Suppose that $P_n \in \mathcal{P}_n$ satisfies*

$$\|(g - P_n)\, w\|_{L_p[-1,1]} \leqslant L\omega_{r,p}\left(g, w, \frac{1}{n}\right). \tag{5.8}$$

Then

$$\|P_n^{(r)} w\Phi_{1/n}^r\|_{L_p[-1,1]} \leqslant Cn^r \omega_{r,p}\left(g, w, \frac{1}{n}\right), \tag{5.9}$$

where $C \neq C(g, n)$.

Proof. Choose P_n^* such that

$$\|(g - P_n^*)\, w\|_{L_p[-1,1]} + n^{-r}\, \|P_n^{*(r)} w\Phi_{1/n}^r\|_{L_p[-1,1]}$$
$$\leqslant 2\bar{K}_{r,p}(g, w, n^{-r}) \leqslant C_2 \omega_{r,p}\left(g, w, \frac{1}{n}\right).$$

Then

$$\|(P_n - P_n^*)\, w\|_{L_p[-1,1]} \leqslant C_3 \omega_{r,p}\left(g, w, \frac{1}{n}\right).$$

From the Markov–Bernstein inequality (2.14), we deduce that

$$\|(P_n - P_n^*)^{(r)}\, w\Phi_{1/n}^r\|_{L_p[-1,1]} \leqslant C_4 n^r \omega_{r,p}\left(g, w, \frac{1}{n}\right).$$

Then

$$\|P_n^{(r)} w\Phi_{1/n}^r\|_{L_p[-1,1]} \leqslant C_5 \{ \|P_n^{*(r)} w\Phi_{1/n}^r\|_{L_p[-1,1]}$$
$$+ \|(P_n - P_n^*)^{(r)}\, w\Phi_{1/n}^r\|_{L_p[-1,1]}\}$$
$$\leqslant C_6 n^r \omega_{r,p}\left(g, w, \frac{1}{n}\right). \quad \blacksquare$$

Finally, we give the

Proof of Corollary 1.6(b). It is clear that

$$\bar{K}_{r,p}(f, w, t^r) \geqslant K_{r,p}(f, w, t^r).$$

To prove the converse inequality, choose g such that

$$\|(f - g)\, w\|_{L_p[-1,1]} + t^r\, \|g^{(r)} w\Phi_t^r\|_{L_p[-1,1]} \leqslant 2K_{r,p}(f, w, t^r).$$

Let $n := \langle 1/t \rangle$. By Corollary 1.4, we can choose $P_n \in \mathscr{P}_n$ such that

$$\|(g-P_n)\, w\|_{L_p[-1,1]} \leqslant C_2 \omega_{r,p}\left(g, w, \frac{1}{n}\right).$$

Then by the lemma above,

$$n^{-r}\, \|P_n^{(r)} w \Phi_{1/n}^r\|_{L_p[-1,1]} \leqslant C_2 \omega_{r,p}\left(g, w, \frac{1}{n}\right).$$

Then as $\Phi_t \sim \Phi_{1/n}$, we obtain (using (1.17), (1.19))

$$\begin{aligned}
\bar{K}_{r,p}(f, w, t^r) &\leqslant C_3 \bar{K}_{r,p}\left(f, w, \frac{1}{n^r}\right)\\
&\leqslant C_3\{\|(f-P_n)\, w\|_{L_p[-1,1]} + n^{-r}\, \|P_n^{(r)} w\Phi_{1/n}^r\|_{L_p[-1,1]}\}\\
&\leqslant C_3\{\|(f-g)\, w\|_{L_p[-1,1]} + \|(g-P_n)\, w\|_{L_p[-1,1]}\\
&\quad + n^{-r}\, \|P_n^{(r)} w\Phi_{1/n}^r\|_{L_p[-1,1]}\}\\
&\leqslant C_4\left\{K_{r,p}(f, w, t^r) + \omega_{r,p}\left(g, w, \frac{1}{n}\right)\right\}\\
&\leqslant C_5\{K_{r,p}(f, w, t^r) + n^{-r}\, \|g^{(r)} w\Phi_{1/n}^r\|_{L_p[-1,1]}\}\\
&\leqslant C_6 K_{r,p}(f, w, t^r)
\end{aligned}$$

by Corollary 1.6(a) and choice of g. ∎

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